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# New two-dimensional integrable quantum models from SUSY intertwining

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**Abstract**
Supersymmetrical intertwining relations of second order in the derivatives are investigated for the case of supercharges with deformed hyperbolic metric $g_{ik} = \mathrm{diag}(1, -a^2)$. Several classes of particular solutions of these relations are found. The corresponding Hamiltonians do not allow the conventional separation of variables, but they commute with symmetry operators of fourth order in momenta. For some of these models the specific SUSY procedure of separation of variables is applied.

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## 1. Introduction

Two-dimensional supersymmetric quantum mechanics [1] with supercharges of second order in derivatives was shown to be instrumental in building new integrable two-dimensional quantum and classical systems which do not allow the standard separation of variables [2–6]. All such systems have, by construction, symmetry operators of fourth order in momenta which, as a rule, are not reducible to lower order. Therefore, these models *do not allow* the conventional separation of variables (for definitions see [7]).

The possible intertwining operator—supercharges—can be easily classified into four inequivalent classes depending on their metrics $g_{ik}$ [4]. The intertwining relations of second order can be represented in the form of a system of six nonlinear partial differential equations involving two partner potentials plus six functions defining the supercharge. It does not seem to be possible to solve this system in a general form. Till now, only some particular cases of constant metric were investigated [2–6]: elliptic (Laplacian) $g_{ik} = \delta_{ik}$, Lorentz

$g_{ik} = \mathrm{diag}(1, -1)$, and degenerate $g_{ik} = \mathrm{diag}(1, 0)$. The elliptic case leads to models with separation of variables and second-order symmetry operators, but the other gave rise to a list of *completely integrable* models (with fourth-order symmetry operators) which are not amenable to separation of variables. Some of these models were found previously by other methods [8], but some of them are new. We notice that although, in general, there is no direct relation between the integrability (with symmetry operators of order higher than 2 in momenta) and the solvability of the system, it was shown that some models with this kind of integrability allow partial [3] or even complete [6] solvability by means of SUSY methods.

Only one class of intertwining supercharges *with constant metric* in the supercharges was not yet investigated: the case of a diagonal matrix $g_{ik} = \mathrm{diag}(1, -a^2), a \neq 0, \pm 1$. Just the study of this kind of intertwining relation is the objective of the present paper, the corresponding metric being naturally called a 'deformed hyperbolic' one.

The paper is organized as follows. In section 2 the solution of the intertwining relations with deformed hyperbolic metrics is performed, in such a way that the unknown functions satisfy a unique nonlinear functional-differential equation. Since the general solution of this equation is very difficult to find, some suitable ansätze are investigated in section 3, and particular solutions of the intertwining relations are obtained explicitly. In section 4 the special case when variables are separated only in one of the partner Hamiltonians is considered. A new SUSY-algorithm of separation of variables for the second partner potential is proposed for this particular class of models. In section 5 we consider the solution of the intertwining relations with deformed hyperbolic metric when one of the partner is chosen to be the isotropic harmonic oscillator. It is shown that its superpartner necessarily allows separation of variables too. Some final conclusions put an end to this paper.

## 2. SUSY intertwining relations for supercharges with deformed hyperbolic metric

Let us consider the SUSY intertwining relations

$$H_1 Q^+ = Q^+ H_2 \qquad Q^- H_1 = H_2 Q^- \tag{1}$$

between two two-dimensional partner Hamiltonians of Schrödinger type

$$H_{1,2} = -\triangle + V_{1,2}(\vec{x}) \qquad \triangle \equiv \partial_1^2 + \partial_2^2 \qquad \partial_i \equiv \partial/\partial x_i \qquad \vec{x} = (x_1, x_2) \tag{2}$$

with second-order supercharges of the form

$$Q^+ = g_{ik}(\vec{x})\partial_i\partial_k + C_i(\vec{x})\partial_i + B(\vec{x}) \qquad Q^- = (Q^+)^\dagger. \tag{3}$$

These intertwining relations realize the isospectrality (up to zero modes of $Q^\pm$) of the superpartners $H_1, H_2$ and the connection between their wavefunctions with the same values of energy:

$$\Psi_n^{(1)}(x) = Q^+\Psi_n^{(2)}(x) \qquad \Psi_n^{(2)}(x) = Q^-\Psi_n^{(1)}(x) \qquad n = 0, 1, 2, \ldots. \tag{4}$$

Equations (1) are equivalent [2] to the following system of six nonlinear partial differential equations

$$\partial_i C_k + \partial_k C_i - (V_1 - V_2)g_{ik} = 0 \tag{5}$$

$$\triangle C_i + 2\partial_i B + 2g_{ik}\partial_k V_2 - (V_1 - V_2)C_i = 0 \tag{6}$$

$$\triangle B + g_{ik}\partial_i\partial_k V_2 + C_i\partial_i V_2 - (V_1 - V_2)B = 0 \tag{7}$$

where, for simplicity, the explicit dependence of the functions on the variables has been eliminated.

For the particular case $g_{ik} = \mathrm{diag}(1, -a^2)$, $a \neq 0, \pm 1$, considered in this paper, the three equations in (5) take the form:

$$2\partial_1 C_1 = v \qquad 2\partial_2 C_2 = -a^2 v \tag{8}$$

$$\partial_1 C_2 + \partial_2 C_1 = 0 \tag{9}$$

where the notation $v(\vec{x}) \equiv V_1(\vec{x}) - V_2(\vec{x})$ has been introduced. From equation (8) one can easily express the three functions $C_{1,2}$ and $v$ in terms of a unique arbitrary function $C(\vec{x})$:

$$C_1(\vec{x}) = -\frac{1}{a^2}\partial_2 C(\vec{x}) \qquad C_2(\vec{x}) = \partial_1 C(\vec{x}) \qquad v(\vec{x}) = -\frac{2}{a^2}\partial_1\partial_2 C(\vec{x}). \tag{10}$$

Then, due to (9), the arbitrary function $C(\vec{x})$ must satisfy the second-order wave equation:

$$\left(\partial_1^2 - \frac{1}{a^2}\partial_2^2\right)C(\vec{x}) = 0. \tag{11}$$

The new variables $x_\pm \equiv x_1 \pm a x_2$ are obviously suitable to write the general solution of (11) in terms of two arbitrary functions $C_\pm(x_\pm)$ as follows:

$$C(\vec{x}) = \int C_+(x_+)\,\mathrm{d}x_+ + \int C_-(x_-)\,\mathrm{d}x_-. \tag{12}$$

Therefore, from (10) we obtain

$$C_1(\vec{x}) = -\frac{1}{a}(C_+(x_+) - C_-(x_-)) \qquad C_2(\vec{x}) = C_+(x_+) + C_-(x_-)$$
$$v(\vec{x}) = -\frac{2}{a}(C'_+(x_+) - C'_-(x_-)) \tag{13}$$

where the prime means derivative of the corresponding function with respect to its argument (we will also use below the notation $\partial_\pm \equiv \partial/\partial x_\pm$).

The two equations in (6) can be rewritten now in terms of the functions $C_+, C_-$ and their derivatives as follows:

$$-\frac{1+a^2}{a}(C''_+ - C''_-) + 2(\partial_+ + \partial_-)(B + V_2) - \frac{2}{a^2}(C'_+ - C'_-)(C_+ - C_-) = 0 \tag{14}$$

$$\frac{1+a^2}{a}(C''_+ + C''_-) + 2(\partial_+ - \partial_-)(B - a^2 V_2) + \frac{2}{a^2}(C'_+ - C'_-)(C_+ + C_-) = 0. \tag{15}$$

Simple linear combinations of (14) and (15) lead to the system:

$$a\partial_+(2B + (1-a^2)V_2) = -\frac{2}{a}(C'_+ - C'_-)C_-(1+a^2)C''_- - a(1+a^2)\partial_- V_2 \tag{16}$$

$$a\partial_-(2B + (1-a^2)V_2) = \frac{2}{a}(C'_+ - C'_-)C_+ + (1+a^2)C''_+ - a(1+a^2)\partial_+ V_2. \tag{17}$$

The consistency condition for these two equations is

$$\partial_1\partial_2\left[a^2(1+a^2)V_2 - \left(C_+^2 + C_-^2\right) - a(1+a^2)(C'_+ - C'_-)\right] = 0. \tag{18}$$

Therefore, the partner Hamiltonians can be expressed in terms of the four arbitrary functions $C_+(x_+)$, $C_-(x_-)$, $F_1(x_1)$ and $F_2(x_2)$:

$$H_{1,2} = -\Delta + V_{1,2}(\vec{x}) \qquad \Delta = \partial_1^2 + \partial_2^2 = (1+a^2)\left(\partial_+^2 + \partial_-^2\right) + 2(1-a^2)\partial_+\partial_- \tag{19}$$

$$V_{1,2}(\vec{x}) = \mp\frac{1}{a}(C_+'(x_+) - C_-'(x_-)) + \frac{1}{a^2(1+a^2)}\big(C_+^2(x_+) + C_-^2(x_-)\big) + F_1(x_1) + F_2(x_2). \quad (20)$$

From (16), (17) and (20), one obtains the expression for $B$ :

$$B(\vec{x}) = -\frac{1}{a^2}C_+(x_+)C_-(x_-) - \frac{1-a^2}{2a}(C_+'(x_+) - C_-'(x_-)) - \frac{1-a^2}{2a^2(1+a^2)}\big(C_+^2(x_+) + C_-^2(x_-)\big) - (F_1(x_1) - a^2F_2(x_2)). \quad (21)$$

Thus, all these algebraic manipulations transformed the initial problem of solving the system of differential equations (5)–(7) to expressions (13), (20) and (21) in terms of arbitrary functions $C_+(x_+)$, $C_-(x_-)$, $F_1(x_1)$, $F_2(x_2)$, which are restricted by the only remaining equation (7):

$$\partial_+\big\{bC_+''(x_+) + 2C_+(x_+)\big[C_+^2(x_+) + C_-^2(x_-) + 2F(\vec{x})\big]\big\} = \partial_-\big\{bC_-''(x_-) + 2C_-(x_-)\big[C_+^2(x_+) + C_-^2(x_-) + 2F(\vec{x})\big]\big\} \quad (22)$$

where the function $F$ and the constant $b$ were defined as

$$F(\vec{x}) \equiv \frac{a^2(1+a^2)}{1-a^2}(F_1(x_1) - a^2F_2(x_2)) \qquad b \equiv -a^2(1+a^2)^2. \quad (23)$$

The connection with previous works on the Lorentz metric case [2–6] can be established very easily. Indeed, equation (22) should be multiplied by $(1-a^2)$, and the limit $a^2 \to 1$ should be taken. Then, the result given in equation (13) of [3] is straightforwardly obtained.

Since equation (22) is the functional-differential equations for the functions $C_\pm$, $F_{1,2}$, which depend on their own arguments, there are no chances to solve this equation in the most general form. Nevertheless, in the next section some particular solutions of (22) will be found by suitably chosen ansätze. In the following we are mainly interested in the models where at least one of the partner Hamiltonians is not amenable to separation of variables.

## 3. Particular solutions of intertwining relations

In order to obtain particular solutions of the functional-differential equation (22), some extra hypothesis must be done. We will consider the following three cases.

### *3.1. Case in which $C_-(x_-) = 0$*

A great simplification is obtained if we choose one of the functions $C_\pm$ to be zero. Without loss of generality, let us consider $C_-(x_-) = 0$. Then, one has to solve the second-order equation for the functions $C_+(x_+)$, $F(\vec{x})$ :

$$bC_+''(x_+) + 2C_+^3(x_+) + 4C_+(x_+)F(\vec{x}) = U_-(x_-) \quad (24)$$

where $U_-$ is an arbitrary function. Different possibilities appear here:

(1) $U_-(x_-) = \text{const}$
From (24), this choice gives immediately that $F(\vec{x})$ actually depends only on $x_+$. Therefore, from (23), $F_{1,2}(x_{1,2})$ are at most linear functions of the corresponding arguments, and $F(\vec{x}) = c_+x_+ + c_0$. This form of $F(\vec{x})$ allows separation of variables for both Hamiltonians (19): to separate variables one has to rewrite (19) in terms of $z_+ = x_+ = x_1 + ax_2$ and $z_- = ax_1 - x_2$. In the variables $z_\pm$ the kinetic part of the Hamiltonian, in contrast to variables $x_\pm$ (see (19)), does not contain mixed terms, and the separation of variables for a linear function $F$ becomes evident.

(2) $U_-(x_-) \neq \text{const}$
In this case, from equation (24), one can express $F(\vec{x})$ as

$$4F(\vec{x}) = U_+(x_+)U_-(x_-) - L_+(x_+) \tag{25}$$

where the new functions are defined as

$$U_+(x_+) \equiv C_+^{-1}(x_+) \qquad L_+(x_+) \equiv \frac{bC_+''(x_+) + 2C_+^3(x_+)}{C_+(x_+)} = \frac{2[1 + b(U_+')^2]}{U_+^2} - \frac{bU_+''}{U_+}. \tag{26}$$

The special form (23) of the function $F(\vec{x})$ means that

$$\begin{gathered}\left(\partial_+^2 - \partial_-^2\right)F(\vec{x}) = U_+''(x_+)U_-(x_-) - U_+(x_+)U_-''(x_-) - L_+''(x_+) = 0 \\ \frac{U_+''(x_+)}{U_+(x_+)} = \frac{U_-'''(x_-)}{U_-'(x_-)} \equiv \eta^2\end{gathered} \tag{27}$$

where $\eta$ is an arbitrary *real or purely imaginary* constant.
For $\eta = 0$, the functions $U_+(x_+), U_-(x_-)$ are polynomials of first and second order, respectively. After their substitution into (27), and comparing with (26), one can check that both partner Hamiltonians allow separation, either in the variables $x_\pm$ or in the variables $z_\pm$.

For $\eta \neq 0$, the generic form of the functions $U_\pm$ is

$$U_+(x_+) = \sigma_+ \exp(\eta x_+) + \delta_+ \exp(-\eta x_+) \tag{28}$$

$$U_-(x_-) = \sigma_- \exp(\eta x_-) + \delta_- \exp(-\eta x_-) + \delta \tag{29}$$

where the constant coefficients $\sigma_\pm, \delta_\pm$ must assure the real character of $U_\pm(x_\pm)$. Again, comparing (27) and (26), after a simple calculation one can obtain that $L_+ = b\eta^2\delta = 0$; $b\sigma_+\delta_+\eta^2 = 1/4$, and

$$F_1(x_1) = \frac{1 - a^2}{4a^2(1 + a^2)}(\sigma_+\sigma_- \exp(+2\eta x_1) + \delta_+\delta_- \exp(-2\eta x_1)) + k_1 \tag{30}$$

$$F_2(x_2) = -\frac{1 - a^2}{4a^4(1 + a^2)}(\sigma_+\delta_- \exp(+2a\eta x_2) + \delta_+\sigma_- \exp(-2a\eta x_2)) + k_2 \tag{31}$$

where the constants $k_i$ are such that $k_1 - a^2k_2 = (1 - a^4)\eta^2/4$. The expressions (20) for the potentials $V_{1,2}$ include also the function $C_+(x_+)$, which has to be found from (26), i.e.

$$bC_+''(x_+) + 2C_+^3(x_+) - b\eta^2C_+(x_+) = 0. \tag{32}$$

This equation can be integrated once:

$$b(C_+'(x_+))^2 = -\left(C_+^2(x_+) - b\eta^2/2\right)^2 + C \qquad C = \text{const}. \tag{33}$$

For the arbitrary value of the real constant $C$ the function $C_+(x_+)$ can be expressed in terms of elliptic functions (see [9]), and the corresponding potentials in (20) will be written in terms of such $C_+$ and $F$ from (30), (31). Both partner Hamiltonians do not allow separation of variables.

For the specific value $C = 0$, equation (33) becomes a couple of Ricatti equations [10] (let us remind that by definition $b = -a^2(1 + a^2)^2 < 0$):

$$\sqrt{-b}C_+'(x_+) = \pm\left(C_+^2(x_+) - b\eta^2/2\right). \tag{34}$$

The solutions are well known:

$$C_+(x_+) = \mp\frac{\mathrm{i}\eta\sqrt{-b}}{\sqrt{2}}\frac{\nu \exp(\mathrm{i}\eta x_+/\sqrt{2}) - \tilde{\nu} \exp(-\mathrm{i}\eta x_+/\sqrt{2})}{\nu \exp(\mathrm{i}\eta x_+/\sqrt{2}) + \tilde{\nu} \exp(-\mathrm{i}\eta x_+/\sqrt{2})} \tag{35}$$

where the constants $\nu$, $\tilde{\nu}$ have to keep $C_+(x_+)$ real.

Choosing the lower sign in (35) one obtains from (20), (30) and (31) the following expressions for the partner potentials (up to a common constant term):

$$V_2(\vec{x}) = F_1(x_1) + F_2(x_2) - (1+a^2)\eta^2/2 = \frac{1-a^2}{4a^4(1+a^2)} \times [a^2(\sigma_+\sigma_-\,\mathrm{e}^{2\eta x_1} + \delta_+\delta_-\,\mathrm{e}^{-2\eta x_1}) - (\sigma_+\delta_-\mathrm{e}^{2a\eta x_2} + \delta_+\sigma_-\,\mathrm{e}^{-2a\eta x_2})] \tag{36}$$

$$V_1(\vec{x}) = V_2(\vec{x}) - \frac{2}{a}C_+'(x_+) = V_2(\vec{x}) + \frac{2}{a^2(1+a^2)}C_+^2(x_+) + \eta^2. \tag{37}$$

Choosing the other sign in (35) is equivalent to the interchange $V_1 \leftrightarrow V_2$. The constants in potential $V_1$ can be tuned so that it grows at $|\vec{x}| \to \infty$, and the corresponding Schrödinger equation will allow separation in terms of the variables $x_1, x_2$. The solutions of both one-dimensional equations can be represented in terms of Mathieu functions [9]. Therefore, the Schrödinger equation with potential $V_1(\vec{x})$ in (37), which does not allow separation of variables, is isospectral (up to zero modes of supercharges) to the Schrödinger equation with potential $V_2$ in (36). Since this last potential allows obviously the separation of variables, one obtains a specific SUSY-separation of variables for potential (37). For further developments of this model, see section 4 below.

### *3.2. Two Ricatti equations*

The second ansatz which will give some solutions of the intertwining relations, i.e. solutions of (22), is

$$C_\pm'(x_\pm) = cC_\pm^2(x_\pm) + d \tag{38}$$

with $c, d$ being arbitrary non-zero real constants. The general solutions of these Ricatti equations are

$$C_\pm = d\cdot\frac{f_\pm}{f_\pm'} \qquad f_\pm = \sigma_\pm\exp(\gamma x_\pm) + \delta_\pm\exp(-\gamma x_\pm) \qquad \gamma \equiv \sqrt{-cd} \tag{39}$$

where we can have $cd > 0$ or $cd < 0$.

Substituting relations (38) into (22), one can check that this last equation takes a simple form if $ca(1+a^2) = \pm 1$, indeed:

$$\partial_+(C_+(x_+)(F(\vec{x}) - d/c)) = \partial_-(C_-(x_-)(F(\vec{x}) - d/c)) \tag{40}$$

with $F(\vec{x})$ defined in (23). This partial differential equation for $F$ can be solved in a general form (see [5]) by introducing the new variables $t_\pm \equiv \int \mathrm{d}x_\pm/C_\pm(x_\pm)$ :

$$\partial_{t_+}(C_+C_-(F - d/c)) = \partial_{t_-}(C_+C_-(F - d/c)). \tag{41}$$

Therefore, its general solution is expressed in terms of an arbitrary function $M$ of the combination $(t_+ + t_-)$. Hence, we have

$$F(\vec{x}) - d/c = M\left(\int\frac{\mathrm{d}x_+}{C_+} + \int\frac{\mathrm{d}x_-}{C_-}\right)\Big/ C_+(x_+)C_-(x_-) \tag{42}$$

where both functions $C_\pm(x_\pm)$ are given by (39). Then, equation (42) reads

$$F(\vec{x}) - d/c = \frac{1}{d^2}M\left(\frac{1}{d}\ln(f_+f_-)\right)f_+'f_-'/f_+f_- \equiv U(f_+f_-)f_+'f_-'. \tag{43}$$

Now one has to remind the special dependence of $F(\vec{x})$ on $\vec{x}$ given in (23), i.e. $\partial_1\partial_2F(\vec{x}) = 0$, which means that $U''(f_+f_-) = 0$. Finally, we obtain the following solutions for $F_{1,2}$ :

$$F_1(x_1) = k_1(\sigma_+\sigma_-\,\mathrm{e}^{2\gamma x_1} + \delta_+\delta_-\,\mathrm{e}^{-2\gamma x_1}) + k_2\big(\sigma_+^2\sigma_-^2\,\mathrm{e}^{4\gamma x_1} + \delta_+^2\delta_-^2\,\mathrm{e}^{-4\gamma x_1}\big) + c_1$$

$$F_2(x_2) = \frac{k_1}{a^2}(\sigma_+\delta_-\,\mathrm{e}^{2a\gamma x_2} + \sigma_-\delta_+\,\mathrm{e}^{-2a\gamma x_2}) + \frac{k_2}{a^2}\big(\sigma_+^2\delta_-^2\,\mathrm{e}^{4a\gamma x_2} + \sigma_-^2\delta_+^2\,\mathrm{e}^{-4a\gamma x_2}\big) + c_2$$

where $c_1 - a^2c_2 = cd(1-a^4)$. If we choose $c = +1/a(1+a^2)$, the partner potentials can be written in an explicit form

$$V_1 = -\frac{8d\sigma_-\delta_-}{a(\sigma_-\,\mathrm{e}^{\gamma x_-} - \delta_-\,\mathrm{e}^{-\gamma x_-})^2} + k_1(\sigma_+\sigma_-\,\mathrm{e}^{2\gamma x_1} + \delta_+\delta_-\,\mathrm{e}^{-2\gamma x_1}) + k_2\big(\sigma_+^2\sigma_-^2\,\mathrm{e}^{4\gamma x_1} + \delta_+^2\delta_-^2\,\mathrm{e}^{-4\gamma x_1}\big) + \frac{k_1}{a^2}(\sigma_+\delta_-\,\mathrm{e}^{2a\gamma x_2} + \sigma_-\delta_+\,\mathrm{e}^{-2a\gamma x_2}) + \frac{k_2}{a^2}\big(\sigma_+^2\delta_-^2\,\mathrm{e}^{4a\gamma x_2} + \sigma_-^2\delta_+^2\,\mathrm{e}^{-4a\gamma x_2}\big) \qquad (44)$$

$$V_2 = -\frac{8d\sigma_+\delta_+}{a(\sigma_+\,\mathrm{e}^{\gamma x_+} - \delta_+\,\mathrm{e}^{-\gamma x_+})^2} + k_1(\sigma_+\sigma_-\,\mathrm{e}^{2\gamma x_1} + \delta_+\delta_-\,\mathrm{e}^{-2\gamma x_1)}) + k_2\big(\sigma_+^2\sigma_-^2\,\mathrm{e}^{4\gamma x_1} + \delta_+^2\delta_-^2\,\mathrm{e}^{-4\gamma x_1}\big) + \frac{k_1}{a^2}(\sigma_+\delta_-\,\mathrm{e}^{2a\gamma x_2} + \sigma_-\delta_+\,\mathrm{e}^{-2a\gamma x_2}) + \frac{k_2}{a^2}\big(\sigma_+^2\delta_-^2\,\mathrm{e}^{4a\gamma x_2} + \sigma_-^2\delta_+^2\,\mathrm{e}^{-4a\gamma x_2}\big) \qquad (45)$$

up to a common additive constant. The alternative choice $c = -1/a(1+a^2)$ corresponds to the interchange $V_1 \leftrightarrow V_2$.

We can observe that by choosing one of the following constants $\{\delta_+, \sigma_+, \delta_-, \sigma_-\}$ to be zero, we will obtain that only one of the partner Hamiltonians $H_1$, $H_2$ admits separation of variables. For example, $\sigma_+ = 0$ allows us to separate variables in $H_2$ (but not in $H_1$); the two one-dimensional potentials, that appear after separation of variables, are exactly solvable Morse potentials, and their eigenfunctions are known analytically. The separation of variables in $H_2$ gives a chance to investigate a whole variety of bound states for the partner Hamiltonian $H_1$ too. In this sense one can speak of a specific kind of 'SUSY-separation of variables' in $H_1$ (for other types of SUSY-separation of variables see also [3, 6] and section 4 below).

### 3.3. Case in which $C'_\pm(x_\pm) = cC^2_\pm(x_\pm)$

This ansatz is a limit case of the previous one, (38) with $d = 0$. The corresponding solutions are

$$C_\pm(x_\pm) = -1/cx_\pm. \qquad (46)$$

A calculation similar to the one carried out in the previous subsection gives us

$$F(\vec{x}) = (x_+x_-)M\big(x_+^2 + x_-^2\big) \qquad M'' = 0 \qquad (47)$$

$$F_1(x_1) = a_1x_1^2 + b_1x_1^4 \qquad F_2(x_2) = a_1x_2^2 + b_1a^2x_2^4 \qquad (48)$$

$$V_{1,2}(\vec{x}) = \frac{2(1+a^2)}{(x_\pm)^2} + a_1\big(x_1^2 + x_2^2\big) + b_1\big(x_1^4 + x_2^4\big). \qquad (49)$$

The two isospectral Hamiltonians with potentials (49) do not admit separation of variables[4].

[4] Due to the coefficients of the attractive singular terms in (49), these Hamiltonians are symmetric operators, but they have no self-adjoint extensions (see [11]).

## 4. SUSY-separation of variables

As we noticed in the examples worked out in the previous section, there are some models for which one of the partner Hamiltonians allows separation of variables, but the other does not. This means that the solution of the two-dimensional Schrödinger equation for the separable Hamiltonian (for example, $H_2$) is reduced to the solution of two separate one-dimensional Schrödinger problems. Its wavefunctions are expressed as a bilinear combination of these one-dimensional wavefunctions with arbitrary constant coefficients. We can use the SUSY intertwining relations (4) to obtain the wavefunctions $\Psi_n^{(1)}$ of $H_1$, one can use $\Psi_n^{(1)}$ from the now known wavefunctions $\Psi_n^{(2)}$, acting with the supercharge $Q^+$. But in contrast to the one-dimensional situation, some additional eigenstates of $H_1$ may exist: if they are annihilated by $Q^-$, there are no partners in the spectrum of $H_2$. Therefore, our task now is to find all the normalizable wavefunctions of $H_1$, which are simultaneously the zero modes of $Q^-$.

### *4.1. The algorithm*

To resolve this problem in the case of separation of variables in $H_2$, we will use the following trick: though the variables are separated neither in $H_1$, nor in $Q^-$, we will consider a linear combination:

$$Z \equiv \alpha H_1 + \beta Q^- \qquad \alpha, \beta = \text{const}. \tag{50}$$

Let us suppose now the existence of some constants $\alpha$, $\beta$, such that the variables in the operator $Z$ are separated by some similarity transformation, and its normalizable eigenfunctions could be found. Then, one has to extract, among these eigenfunctions (by the direct action of $Q^-$), those which are simultaneously the normalizable zero modes of $Q^-$. If this plan can be realized, we will reduce the spectral problem for the Hamiltonian $H_1$ (which is not amenable to conventional separation of variables) to a couple of one-dimensional spectral problems.

Keeping in mind [3] that sometimes the preliminary similarity transformations are helpful for separation of variables, we will transform the operator $Z$ using a function $\exp\{\varphi(\vec{x})\}$ that will be determined later on:

$$\begin{aligned} Y \equiv \mathrm{e}^{-\varphi(\vec{x})} Z\, \mathrm{e}^{\varphi(\vec{x})} &= (\beta-\alpha)\partial_1^2 - (\alpha+\beta a^2)\partial_2^2 + 2(\beta-\alpha)(\partial_1\varphi)\partial_1 \\ &\quad - 2(\alpha+\beta a^2)(\partial_2\varphi)\partial_2 - \beta C_k\partial_k + (\beta-\alpha)\big(\big(\partial_1^2\varphi\big) + (\partial_1\varphi)^2\big) \\ &\quad - (\alpha+\beta a^2)\big(\big(\partial_2^2\varphi\big) + (\partial_2\varphi)^2\big) - \beta C_k\partial_k\varphi + \beta(B - \partial_k C_k) + bV_1. \end{aligned} \tag{51}$$

To exclude the first-order derivatives in $Y$, we have to impose two conditions:

$$2(\beta-\alpha)(\partial_1\varphi) = \beta C_1 \qquad -2(\alpha+\beta a^2)(\partial_2\varphi) = \beta C_2 \tag{52}$$

which are satisfied if

$$(\alpha+\beta a^2)\partial_2 C_1 = -(\beta-\alpha)\partial_1 C_2. \tag{53}$$

Comparing with (9), it is clear that the two constants introduced in (50) are related by $2\alpha = \beta(1-a^2)$.

Substituting now $\partial_1\varphi$, $\partial_2\varphi$ from (52) into (51), we get the expression:

$$Y = \frac{\beta(1+a^2)}{2}\left(\partial_1^2 - \partial_2^2 + \frac{\partial_k C_k}{1+a^2} - \frac{C_1^2 - C_2^2}{(1+a^2)^2}\right) + \beta\left(B - \partial_k C_k + \frac{1-a^2}{2}V_1\right). \tag{54}$$

Taking into account that from (8) $\partial_k C_k = (1-a^2)(V_1 - V_2)/2$ and also (21), one obtains

$$B - \partial_k C_k + \frac{1-a^2}{2}V_1 = B + \frac{1-a^2}{2}V_2 = -\frac{1}{a^2}C_+C_- + \frac{1+a^2}{2}(F_2 - F_1).$$

Now using (13), we can write (54) in the form:

$$Y = \frac{\beta(1+a^2)}{2}\left(\partial_1^2 - \partial_2^2 - \frac{1-a^2}{a(1+a^2)}\left(C_+' + \frac{C_+^2}{a(1+a^2)} - C_-' + \frac{C_-^2}{a(1+a^2)}\right) + F_2 - F_1\right). \tag{55}$$

Using the expressions (10)–(13) for $C_i$, the function $\varphi(\vec{x})$ can be written explicitly in terms of $C_\pm(x_\pm)$:

$$\varphi(\vec{x}) = \frac{-1}{a(1+a^2)}\left(\int C_+(x_+)\,\mathrm{d}x_+ - \int C_-(x_-)\,\mathrm{d}x_-\right). \tag{56}$$

Though for generic functions $C_\pm(x_\pm)$ variables in (55) cannot be separated, for some of ansätze considered before this is just possible due to a specific choice of $C_\pm$. In particular, the second ansatz of section 3.1 with $\eta \neq 0, C_- = 0$ and $C_+$ satisfying (34), eliminates all obstacles to separate variables for (55) in terms of $x_1, x_2$. The same happens for the ansatz of section 3.2, if one of the constants $\sigma_+, \delta_+$ vanishes. In this case the functions $C_\pm(x_\pm)$ satisfy also the Ricatti equation (38), and the potential $V_2$ in (45) allows separation of variables.

In both of these models, the similarity transformation above separates variables, and the eigenfunctions of the operator $Y$ (and therefore, of $Z$) can be built from the eigenfunctions of the corresponding pair of one-dimensional problems.

Now, we will briefly consider the particular case of the model with partner potentials (44), (45) for $\sigma_+ = 0$. Then, the physical system is described as

$$H_2 = h_1(x_1) + h_2(x_2) \qquad h_1 = -\partial_1^2 + F_1(x_1) \qquad h_2 = -\partial_2^2 + F_2(x_2) \tag{57}$$

$$h_1\psi_1(x_1) = \epsilon_1\psi_1(x_1) \qquad h_2\psi_2(x_2) = \epsilon_2\psi_2(x_2) \tag{58}$$

where the two one-dimensional Hamiltonians $h_1$ and $h_2$ are, up to additive constants $c_1, c_2$, Morse potentials with well-known bound states if $\sigma_-\delta_- > 0$. For simplicity, we will consider $\sigma_- = \delta_- \equiv \sigma$ and $d < 0$. Substituting (39) into (13) and (21) one obtains the expression for the supercharge $Q^+$ :

$$Q^+ = -h_1 + a^2 h_2 + C_i\partial_i + \frac{\sqrt{-da(1+a^2)}}{a^2}C_- + \frac{(1-a^2)d}{a}.$$

The action of this operator on the wavefunctions $\Psi(\vec{x}) = \psi_1(x_1)\psi_2(x_2)$ of $H_2$ will give the wavefunctions (if normalizable) of the partner $H_1$:

$$Q^+\Psi(\vec{x}) = \left(a^2\epsilon_2 - \epsilon_1 + \frac{(1-a^2)d}{a}\right)\Psi(\vec{x}) - \frac{\sqrt{-da(1+a^2)}}{a}(\partial_1 - a\partial_2)\Psi(\vec{x})$$
$$+ \frac{C_-(x_-)}{a}\left(\partial_1 + a\partial_2 - \frac{\sqrt{-da(1+a^2)}}{a}\right)\Psi(\vec{x}). \tag{59}$$

The one-dimensional wavefunctions $\psi_1(x_1)$ and $\psi_2(x_2)$ are known explicitly (see, for example [3]) in terms of hypergeometric functions. One can straightforwardly check that the singularity of $C_-(x_-)$ at $x_- = 0$ in the last term of (59) cannot be compensated. Therefore, the bound states $\Psi(\vec{x})$ of $H_2$ have no normalizable partner states in the spectrum of $H_1$. According to the discussion carried out in the first part of this section, the additional bound states of $H_1$ could be found among normalizable zero modes of $Q^-$. In the considered model, the operator (55) takes a simple form (up to a constant factor and a constant additive term), which allows the separation of variables:

$$Y \sim -h_1(x_1) + h_2(x_2) + \frac{2(1-a^2)}{a(1+a^2)}. \tag{60}$$

Then, the eigenfunctions $\Psi_Z(\vec{x})$ of the operator $Z$, defined in (50), are

$$\Psi_Z = \mathrm{e}^{\varphi(\vec{x})}\Psi_Y(\vec{x}) = \mathrm{e}^{\varphi(\vec{x})} \cdot \psi_1(x_1)\psi_2(x_2) \tag{61}$$

where $\psi_1(x_1)$, $\psi_2(x_2)$ are arbitrary eigenfunctions of $h_1, h_2$, respectively, and the similarity transformation is performed by

$$\mathrm{e}^{\varphi(\vec{x})} \sim \frac{\exp(-\gamma x_+)}{\sinh(\gamma x_-)}. \tag{62}$$

The singularity of (62) at $x_- = 0$ is responsible of the non-normalizability of all eigenfunctions $\Psi_Z(\vec{x})$, including possible zero modes of $Q^-$. Therefore, the Hamiltonian $H_1$ has no bound states at all, despite the presence of two Morse potentials in (44).

## 5. Two-dimensional harmonic oscillator

In the previous successful attempts [2–6] to solve the two-dimensional intertwining relations of second order (1), the system of nonlinear differential equations (5)–(7) was solved step by step, and only at the very end some specific expressions for potentials were obtained. Up to now, it has been impossible to solve these equations straightforwardly, starting from the fixed form of one of the partner potentials, apart from some very simple examples with separation of variables in both $V_1$ and $V_2$. The solution of the problem is not so evident in the case of supercharges with deformed hyperbolic metric. In this section, we will take $V_2(\vec{x})$, as an isotropic harmonic oscillator:

$$V_2(\vec{x}) = \Omega\big(x_1^2 + x_2^2\big) + v_0 = \omega\big(x_+^2 + x_-^2\big) + 2\mu x_+x_- + v_0$$
$$x_\pm = x_1 \pm a x_2 \qquad \omega = \frac{\Omega(1+a^2)}{4a^2} \qquad \mu = -\frac{\Omega(1-a^2)}{4a^2}. \tag{63}$$

One can check that a non-isotropic oscillator does not produce any solution of the intertwining relations. Substituting this potential into (18), we obtain

$$\partial_+^2\big[C_+^2 + \sqrt{-b}C_+'\big] = \partial_-^2\big[C_-^2 - \sqrt{-b}C_-'\big] \equiv 2A \qquad A = \text{const} \tag{64}$$

where $A$ is an arbitrary constant, and $b$ was defined in (23). The variables are separated, and the functions $C_\pm(x_\pm)$ satisfy the Ricatti equations:

$$C_+^2(x_+) + \sqrt{-b}C_+'(x_+) = A^2x_+^2 + A_+x_+ + a_+ \tag{65}$$

$$C_-^2(x_-) - \sqrt{-b}C_-'(x_-) = A^2x_-^2 + A_-x_- + a_-. \tag{66}$$

Then, equations (16), (17) give the expression for the function $B(\vec{x})$ in terms of $C_+, C_-$:

$$B(\vec{x}) = -\frac{1}{a^2}C_+C_- - \frac{\mu(1+a^2)+\omega(1-a^2)}{2}\big(x_+^2+x_-^2\big)$$
$$+ \frac{A - a^2(\mu(1-a^2)+\omega(1+a^2))}{a^2}x_+x_- + \rho_+x_+ + \rho_-x_- + \text{const}.$$

After rather cumbersome but straightforward calculations, one can check that the last equation to be solved (7) is fulfilled only if, in addition to (65)–(66), one of the functions $C_+, C_-$ is linear, for example $C_-(x_-) = \sigma x_-$. Then, we finally obtain that

$$V_1(\vec{x}) = V_2(\vec{x}) - \frac{2}{a}C_+'(x_+) + \text{const}. \tag{67}$$

The Schrödinger equation with such potential $V_1$ *does not allow* separation of variables in terms of $x_\pm$ due to the presence of a mixed term in the Laplacian, but it obviously *allows* separation in the variables $z_+ = x_+ = x_1 + ax_2$, $z_- = ax_1 - x_2$, due to the absence of the mixed term. Therefore, no non-trivial isospectral superpartners of the isotropic harmonic oscillator can be built in the framework of second-order intertwining relations for superchrages with constant metric.

## 6. Conclusions

The investigation of intertwining relations (1) with deformed hyperbolic metric $g_{ik} = \mathrm{diag}(1, -a^2)$, $a \neq 0, \pm 1$ in the supercharge operator (3), that has been carried out in this paper, completes the study of second-order intertwining relations with *constant matrix* $g_{ik}$ in two-dimensional quantum mechanics, which was started in [2–6]. In the present case also some particular classes of solutions for the partner potentials were found (see section 3). Among them, there are pairs in which one of the partners allows separation of variables, but the second one does not. A specific procedure of SUSY-separation of variables is proposed for this case (see section 4). In a particular model, this new algorithm led to the conclusion that the system with the attractive potential (44) for $\sigma_+ = 0, \sigma_- = \delta_-$ does not allow any bound states. Although the nonlinear equation (18) is not amenable, as a rule, to solution with a given potential $V_2$ (e.g. for the Coulomb potential), sometimes this is possible. Thus in the case of one partner being harmonic oscillator, it is shown in section 5 that the second one also allows separation of variables. All models considered in the paper are completely integrable, i.e., nontrivial symmetry operators of fourth order in momenta ($R_1 = Q^+Q^-$; $R_2 = Q^-Q^+$) exist for the deformed hyperbolic metric $g_{ik}$ also.

A few additional ansätze could be considered in the same manner. In particular, one can check that the case $C_-(x_-) = \mathrm{const} \neq 0$ (the natural generalization of section 3.1) leads to a particular solution, obtained above within the ansatz in section 3.2. The partner of the constant potential $V_2 = \mathrm{const}$ also allows separation of variables, similar to the case considered in section 5. In this paper we were only interested in real potentials. Some models with complex potentials (see [6, 12]) can be considered if for example we allow purely imaginary $a \neq \pm \mathrm{i}$ or arbitrary values of $\nu$, $\tilde{\nu}$ in (35).

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